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# Orlicz spaces with convexity or concavity constant one

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**Abstract**

We characterize the classes of function and sequence Orlicz spaces that satisfy upper or lower $p$-estimate with constant one. We also present a characterization of $p$-convexity or $p$-concavity with constant one in Orlicz spaces.


*Keywords:* $p$-Convexity (concavity) with constant one; Upper (lower) $p$-estimate with constant one; Orlicz function and sequence spaces; Dilation operator

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## 1. Introduction

Given $0 < p < \infty$, we say that a quasi-normed lattice $(X, \|\cdot\|)$ is *p-convex*, respectively *p-concave* [2,7], if there are constants $M^{(p)}, M_{(p)} < \infty$, such that

$$\left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leqslant M^{(p)} \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}, \tag{1}$$

respectively

$$\left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \leqslant M_{(p)} \left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|, \tag{2}$$

for every choice of vectors $x_1, \ldots, x_n \in X$. The lattice $X$ is said to satisfy an *upper p-estimate*, respectively a *lower p-estimate*, $0 < p < \infty$, if inequality (1), respectively (2), is fulfilled for any choice of disjointly supported elements $x_1, \ldots, x_n$ in $X$. If these inequalities are satisfied with $M^{(p)} = 1$ ($M_{(p)} = 1$), we say that $X$ is $p$-convex ($p$-concave) with constant one (or one-$p$-convex (concave)). Similarly, we say that $X$ satisfies an upper (lower) $p$-estimate with constant one (or one-upper (lower) $p$-estimate) if the appropriate inequalities are satisfied with constant 1.

It is well known that every $p$-convex Banach lattice can be given an equivalent lattice norm that is $p$-convex with constant one (and the same holds for $p$-concavity and upper (lower) $p$-estimates) [7]. Such a renorming is then a starting point for investigation of several geometric properties. Consequently, very often it may be of interest to determine the values of the convexity (concavity) constants, or upper (lower) estimate constants, for particular classes of lattices equipped with their original (quasi-)norms, especially when these constants are equal to one.

Upper and lower estimates (as well as $p$-convexity and $p$-concavity) with constant one have close relations to classical moduli of convexity and smoothness of power type in Banach lattices [7], and to complex moduli of (PL-)convexity [1]. For example (see [7, 1.f.1]), if a Banach lattice is $p$-convex and $q$-concave (for $1 < p \leqslant 2 \leqslant q < \infty$) with constant one then it is uniformly convex and uniformly smooth with the appropriate moduli of power type $p$ and $q$. Also one-$p$-convexity (concavity) properties are applied in studies of contractive projections and isometries [9].

The purpose of this article is to find criteria for one-upper (lower) $p$-estimate as well as one-$p$-convexity (concavity) of Orlicz spaces. Conditions for corresponding properties without control of the constants in quasi-normed Orlicz spaces have been already known for some time [3,4,7]. With this respect Orlicz spaces have more regular behaviour than general Banach lattices, since for instance they are $p$-convex (respectively $p$-concave) if and only if they satisfy an upper (respectively lower) $p$-estimate. However, these results do not provide any control of the convexity and concavity constants.

Analysis of criteria for $p$-convexity and $p$-concavity of $L_\varphi$ from [3,4,7] suggests that these properties depend on the behaviour of the function $\phi(u) = \varphi(u^{1/p})$. These criteria have been expressed in terms of convexity or concavity of some functions equivalent to $\phi$ (where the precise meaning of the word "equivalent" depends on the underlying measure space). In contrast, the description of one-$p$-convexity and the other related notions depends on the function $\phi$ itself.

Thus in particular we show that for a non-atomic measure, the space $L_\varphi$ is one-$p$-convex if and only if $\phi$ is convex (Theorem 5.1). It appears however that upper estimate with constant one requires essentially less than convexity of $\phi$. This yields to introducing a new class of Orlicz functions, discussed in Section 2. In fact, we show (in Corollary 3.3) that in the case of a non-atomic infinite measure, $L_\varphi$ satisfies an one-upper -$p$-estimate if and only if $\phi$ satisfies the following condition $(C)$:

$$\frac{\phi(au)}{\phi(u)} + \frac{\phi((1-a)v)}{\phi(v)} \leqslant 1, \quad \text{for all } u, v > 0 \text{ and } 0 < a < 1.$$

We investigate also the condition on $\phi$, called starshapeness (which means that $\phi(u)/u$ is increasing), which is weaker than convexity but still implies one-upper $p$-estimate of $L_\varphi$. Starshapeness has interesting interpretation in terms of certain inequalities between $f$ and dilation-like element $f^{(\lambda)}$ defined for any measurable function $f$ and any rational number $\lambda$ (Section 4). All of the above characterizations have their counterparts corresponding to concavity and lower estimate with constant one.

As an application of general theorems in Banach lattices and our results, at the end of the paper we formulate sufficient conditions, relatively simple to check, for $L_\varphi$ to have a power type modulus of convexity or smoothness.

The most satisfactory results, stated as necessary and sufficient conditions, are obtained for spaces in the case of infinite non-atomic measures. For spaces in the case of finite non-atomic measure or in the sequence case, the results are similar but often slightly less satisfactory. For instance, in Theorem 3.2 stated for finite non-atomic measures, instead of equivalence of a 1-estimate in $L_\varphi$ and a condition on $\varphi$, as in Theorem 3.1, the condition on $\varphi$ implies the 1-estimate in $L_\varphi$, which in turn implies a restricted form of the condition on $\varphi$.

Let us agree on definitions and further notations appearing in the paper. As usual, by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ we denote the sets of real, non-negative real and natural numbers, respectively. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *Orlicz function* whenever it is strictly increasing and continuous, $\lim_{u\to\infty} \varphi(u) = \infty$, $\varphi(0) = 0$ and $\varphi(1) = 1$. Let $(T, \Sigma, \mu)$ be a $\sigma$-finite measure space and $L^0 \equiv L^0(\mu)$ denote the space of all $\Sigma$-measurable real-valued functions. Given an Orlicz function $\varphi$ we define a *modular* on $L^0$ as follows:

$$I_\varphi(f) = \int_T \varphi\big(\big|f(t)\big|\big)\, d\mu = \int_T \varphi\big(|f|\big).$$

Then the *Orlicz space* $L_\varphi \equiv L_\varphi(T)$ over $(T, \Sigma, \mu)$ is the set of all functions in $L^0$ such that the Minkowski functional of the set $\{f \in L^0\colon\ I_\varphi(f) \leqslant 1\}$ is finite. In other words, $f \in L_\varphi$ whenever

$$\|f\|_\varphi = \inf\big\{\epsilon > 0\colon\ I_\varphi(f/\epsilon) \leqslant 1\big\} < \infty.$$

In the case when $T = \mathbb{N}$ (respectively $T = \{1, \ldots, m\}$, $m \in \mathbb{N}$) and $\mu$ is a counting measure on $2^T$, $L_\varphi$ is an infinite-dimensional sequence Orlicz space further denoted by $\ell_\varphi$ (respectively a finite-dimensional space denoted by $\ell_\varphi^m$). In this case the elements of $\ell_\varphi$ or $\ell_\varphi^m$ are infinite sequences $x = (x_n)$ or finite sequences $x = (x_n)_{n=1}^m$, and the respective modulars have the forms

$$I_\varphi(x) = \sum_{n=1}^{\infty} \varphi(|x_n|) \quad \text{or} \quad I_\varphi(x) = \sum_{n=1}^{m} \varphi(|x_n|).$$

It is well known and easy to show that $L_\varphi$ is a vector space and $\|f\|_\varphi = 0$ if and only if $f = 0$ a.e. in $T$, as well as $\|af\|_\varphi = |a|\|f\|_\varphi$ for all $a \in \mathbb{R}$ and $f \in L_\varphi$ [6,7]. Moreover, if $\varphi$ is convex or concave, then $\|\cdot\|_\varphi$ satisfies the triangle inequality or the reverse triangle inequality, respectively. In particular, if $\varphi$ is convex then $L_\varphi$ equipped with the norm $\|\cdot\|_\varphi$ is a Banach space [5–7]. It is also well known that $\|\cdot\|_\varphi$ is a quasi-norm whenever $\sup_{u>0,\ 0<a<1} \varphi(au)/a^r\varphi(u) < \infty$ for some $r > 0$ [10]. By continuity of $\varphi$ and the Fatou lemma it can be shown that the functional $\|\cdot\|_\varphi$ has the *Fatou property*, that is for any function $f \in L^0$ and any monotone increasing sequence of non-negative measurable functions $(f_n) \subset L_\varphi$ such that $f_n \to f$ a.e. and $\sup_n \|f_n\|_\varphi < \infty$, we have $f \in L_\varphi$ and $\|f_n\|_\varphi \to \|f\|_\varphi$.

We shall study $p$-convexity (concavity), upper (lower) $p$-estimates in $L_\varphi$ with respect to the functional $\|\cdot\|_\varphi$, which might not be even a quasi-norm. We use the analogous definitions of these notions as for quasi-normed spaces $(X, \|\cdot\|)$ simply replacing $\|\cdot\|$ by $\|\cdot\|_\varphi$ in (1) and (2).

By definition of $\|\cdot\|_\varphi$, it is clear that for any $0 < p < \infty$, we have

$$\|f\|_\varphi = \big\||f|^p\big\|_\phi^{1/p} = \big\||f|^p\big\|_{\varphi(u^{1/p})}^{1/p}.$$

Thus $L_\varphi$ is a $p$-convexification of $L_\phi$. It follows that $L_\varphi$ is $p$-convex or $p$-concave with constant one if and only if $L_\phi$ is 1-convex or 1-concave with constant one, respectively. Analogously, $L_\varphi$ satisfies an upper or lower $p$-estimate with constant one if and only if $L_\phi$ satisfies an upper or lower 1-estimate with constant one, respectively. The latter facts allow us later on to reduce the problem of $p$-convexity (concavity) or $p$-estimates of $L_\varphi$ to $p = 1$.

## 2. Classes of Orlicz functions

Let us consider the following classes of Orlicz functions.

Recall that an Orlicz function $\varphi$ is *convex* (respectively *concave*) whenever for every $a \in (0,1)$ and $u, v \in \mathbb{R}_+$, $\varphi(au + (1-a)v) \leqslant a\varphi(u) + (1-a)\varphi(v)$ (respectively $\varphi(au + (1-a)v) \geqslant a\varphi(u) + (1-a)\varphi(v)$).

We say that an Orlicz function $\varphi$ is *starshaped* (respectively *reversed starshaped*) if for all $a \in (0,1)$ and $u \in \mathbb{R}_+$, $\varphi(au) \leqslant a\varphi(u)$ (respectively $\varphi(au) \geqslant a\varphi(u)$).

An Orlicz function $\varphi$ is said to be *superadditive* (respectively *subadditive*) whenever for every $u, v \in \mathbb{R}_+$, $\varphi(u+v) \geqslant \varphi(u) + \varphi(v)$ (respectively $\varphi(u+v) \leqslant \varphi(u) + \varphi(v)$).

It is well known [8] that any Orlicz function which is convex (respectively concave) is also starshaped (respectively reversed starshaped), as well as any Orlicz function which is starshaped (respectively reversed starshaped) is superadditive (respectively subadditive). Neither of these classes coincide.

We introduce here the new class of functions satisfying so called conditions (C) or (RC). We say that $\varphi$ satisfies *condition* (C) (respectively *condition* (RC)) if for every $u, v > 0$ and $0 < a < 1$,

$$\frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)} \leqslant 1 \quad \left(\text{respectively } \frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)} \geqslant 1\right).$$

It is clear that any starshaped (respectively reversed starshaped) Orlicz function $\varphi$ satisfies condition (C) (respectively (RC)), and the latter condition implies that $\varphi$ is superadditive (respectively subadditive). These classes do not coincide as it is shown by the following examples.

**Example 2.1.** Let

$$\varphi(u) = \begin{cases} u^2, & \text{if } 0 < u < 1, \\ 1 + (u-1)^2, & \text{if } u \geqslant 1. \end{cases}$$

Then $\varphi$ is superadditive, but it does not satisfy condition (C).

The first fact is easy to check. For the second one, let us take $a = 1/4$, $u = 2$ and $v = 4/3$. Then we have $(1-a)v = 1$ and so

$$\frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)} = \frac{a^2u^2}{1+(u-1)^2} + \frac{1+((1-a)v-1)^2}{1+(v-1)^2} = \frac{41}{40} > 1.$$

**Example 2.2.** The function

$$\varphi(u) = \begin{cases} u^2, & \text{if } 0 < u < 1, \\ 1 + (u-1)^{3/2}, & \text{if } u \geqslant 1 \end{cases}$$

is not starshaped but it satisfies condition (C).

Indeed, for $u = 3/2$, $a = 2/3$ we have

$$\varphi(au) = 1 > \frac{2}{3} + \frac{1}{3\sqrt{2}} = a\varphi(u).$$

In order to show (C), define the following functions:

$$f(a,u,v) = \frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)}, \quad a \in (0,1),\ u, v \in (0,\infty),$$

$$g(v) = \frac{v^2}{1+(v-1)^{3/2}}, \quad v \in [1,\infty),$$

$$h(a,v) = \frac{1+(av-1)^{3/2}}{1+(v-1)^{3/2}}, \quad a \in (0,1),\ v \in [1/a,\infty).$$

It is easy to show that $g$ is increasing on $[1,\infty)$. Moreover, for fixed $a \in (0,1)$ the function $[1/a,\infty) \ni v \to h(a,v)$ attains its maximum at $1/a$. Indeed, it has just one local extremum on $(1/a,\infty)$, which is a minimum. Moreover,

$$h(a,1/a) = \frac{a^{3/2}}{(1-a)^{3/2} + a^{3/2}} > a^{3/2} = \lim_{v\to\infty} h(a,v).$$

We shall now consider several cases. If $u, v \in (0,1)$ then it is clear that $f$ satisfies condition (C).

Let now $u, (1-a)v \in (0,1)$ and $v \in [1,\infty)$. Since $g$ is increasing, and $v \in [1, 1/(1-a))$ we have

$$f(a,u,v) = a^2 + (1-a)^2 g(v) \leqslant a^2 + \frac{(1-a)^{3/2}}{(1-a)^{3/2} + a^{3/2}} \leqslant 1,$$

since $a^2 \leqslant a^{3/2}/((1-a)^{3/2} + a^{3/2})$ for $a \in (0,1)$.

If $u \in (0,1)$ and $(1-a)v \in [1,\infty)$, then by properties of $h$,

$$\begin{aligned} f(a,u,v) &= a^2 + h(1-a, v) \leqslant a^2 + h\big(1-a, 1/(1-a)\big) \\ &= a^2 + \frac{(1-a)^{3/2}}{(1-a)^{3/2} + a^{3/2}} \leqslant 1. \end{aligned}$$

For $au \in (0,1)$ and $u, (1-a)v \in [1,\infty)$, we have

$$f(a,u,v) = a^2 g(u) + h(1-a, v) \leqslant a^2 g(1/a) + h\big(1-a, 1/(1-a)\big) = 1.$$

Finally, for $au, (1-a)v \in [1,\infty)$, we also have

$$f(a,u,v) = h(a,u) + h(1-a,v) \leqslant h(a, 1/a) + h\big(1-a, 1/(1-a)\big) = 1,$$

which completes the proof that $\varphi$ satisfies (C).

## 3. Upper and lower estimates

Our first result provides a necessary and sufficient condition for upper and lower estimate in $L_\varphi$ with constant one.

**Theorem 3.1.** *Let $\varphi$ be an Orlicz function and $\mu$ a non-atomic infinite measure. The Orlicz space $L_\varphi$ satisfies an upper* 1*-estimate* (*respectively a lower* 1*-estimate*) *with constant one if and only if $\varphi$ satisfies condition* (C) (*respectively* (RC)).

**Proof.** Assume the condition (C) is not satisfied. Thus there exist $u, v > 0$ and $0 < a < 1$ such that

$$\frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)} > 1.$$

Choose disjoint sets $A, B$ satisfying $\mu A = 1/\varphi(u)$ and $\mu B = 1/\varphi(v)$. Letting

$$f = au\chi_A \quad \text{and} \quad g = (1-a)v\chi_B,$$

we have $I_\varphi(f/a) = I_\varphi(g/(1-a)) = 1$, and so $\|f\|_\varphi = a$ and $\|g\|_\varphi = 1-a$. Moreover,

$$\begin{aligned} I_\varphi(f+g) &= I_\varphi(au\chi_A) + I_\varphi\big((1-a)v\chi_B\big) = \varphi(au)\mu A + \varphi\big((1-a)v\big)\mu B \\ &= \frac{\varphi(au)}{\varphi(u)} + \frac{\varphi((1-a)v)}{\varphi(v)} > 1. \end{aligned}$$

It shows that $\|f+g\|_\varphi > 1 = \|f\|_\varphi + \|g\|_\varphi$, and so $L_\varphi$ does not satisfy an upper 1-estimate with constant one.

Assume now the condition (C). We observe first that for any $u_i, w_i > 0$, $i = 1, \dots, n$, $v_j, z_j > 0$, $j = 1, \dots, m$, and $a \in (0,1)$ such that $\sum_{i=1}^n \varphi(u_i)w_i = \sum_{j=1}^m \varphi(v_j)z_j = 1$ we have for some $k = 1, \dots, n$ and $p = 1, \dots, m$,

$$\begin{aligned}
&\sum_{i=1}^n \varphi(au_i)w_i + \sum_{j=1}^m \varphi\big((1-a)v_j\big)z_j \\
&\quad = \sum_{i=1}^n \frac{\varphi(au_i)}{\varphi(u_i)}\varphi(u_i)w_i + \sum_{j=1}^m \frac{\varphi((1-a)v_j)}{\varphi(v_j)}\varphi(v_j)z_j \\
&\quad \leqslant \max_i \frac{\varphi(au_i)}{\varphi(u_i)} \sum_{i=1}^n \varphi(u_i)w_i + \max_j \frac{\varphi((1-a)v_j)}{\varphi(v_j)} \sum_{j=1}^m \varphi(v_j)z_j \\
&\quad = \frac{\varphi(au_k)}{\varphi(u_k)} + \frac{\varphi((1-a)v_p)}{\varphi(v_p)} \leqslant 1.
\end{aligned} \tag{3}$$

Let now $f, g \in L_\varphi$ be non-zero disjoint simple functions. Thus

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad g = \sum_{j=1}^m \beta_j \chi_{B_j}$$

for some mutually disjoint sets $A_i, B_j$ and $\alpha_i, \beta_j \in \mathbb{R}$. Letting $\alpha = \|f\|_\varphi$ and $\beta = \|g\|_\varphi$, we get

$$I_\varphi\left(\frac{f}{\alpha}\right) = \sum_{i=1}^n \varphi\left(\frac{|\alpha_i|}{\alpha}\right)\mu A_i = 1 \quad \text{and} \quad I_\varphi\left(\frac{g}{\beta}\right) = \sum_{j=1}^m \varphi\left(\frac{|\beta_j|}{\beta}\right)\mu B_j = 1.$$

In view of inequality (3) for $a = \alpha/(\alpha+\beta)$, it holds

$$\begin{aligned}
I_\varphi\left(\frac{f+g}{\alpha+\beta}\right) &= I_\varphi\left(a\frac{f}{\alpha}\right) + I_\varphi\left((1-a)\frac{g}{\beta}\right) \\
&= \sum_{i=1}^n \varphi\left(a\frac{|\alpha_i|}{\alpha}\right)\mu A_i + \sum_{j=1}^m \varphi\left((1-a)\frac{|\beta_j|}{\beta}\right)\mu B_j \leqslant 1.
\end{aligned}$$

Therefore $\|f+g\|_\varphi \leqslant \|f\|_\varphi + \|g\|_\varphi$. We finish by the limit argument since $\|\cdot\|_\varphi$ has the Fatou property. □

In a very similar way we can show the result for finite non-atomic measures.

**Theorem 3.2.** *Let $\varphi$ be an Orlicz function and $\mu$ a non-atomic finite measure. If $\varphi$ satisfies condition* (C) (*respectively* (RC)), *then the Orlicz space $L_\varphi$ satisfies an upper* 1*-estimate* (*respectively a lower* 1*-estimate*) *with constant one.*

*Conversely, if $L_\varphi$ satisfies an upper* 1*-estimate* (*respectively a lower* 1*-estimate*) *with constant one, then $\varphi$ satisfies condition* (C) (*respectively* (RC)) *whenever $1/\varphi(u) + 1/\varphi(v) \leqslant \mu T$.*

**Corollary 3.3.** *Let $0 < p < \infty$ and $\mu$ be a non-atomic measure on $T$.*

*If $\mu T = \infty$ then $\varphi(u^{1/p})$ satisfies the condition* (C) (*respectively* (RC)) *if and only if $L_\varphi$ satisfies an upper $p$-estimate* (*respectively a lower $p$-estimate*) *with constant one.*

*Let $\mu T < \infty$. If $\varphi(u^{1/p})$ satisfies condition* (C) (*respectively* (RC)) *then $L_\varphi$ satisfies an upper $p$-estimate* (*respectively a lower $p$-estimate*) *with constant one. Conversely, if $L_\varphi$ satisfies an upper $p$-estimate* (*respectively lower $p$-estimate*) *with constant one, then $\varphi(u^{1/p})$ must satisfy condition* (C) (*respectively* (RC)) *whenever $1/\varphi(u^{1/p}) + 1/\varphi(v^{1/p}) \leqslant \mu T$.*

The next corollary presents relationship between one-$p$-estimates of $L_\varphi$ and starshapeness-conditions of $\varphi$.

**Corollary 3.4.** *Let $0 < p < \infty$ and $\mu$ be a non-atomic measure.*

(i) *If $\varphi(u)/u^p$ is increasing* (*respectively $\varphi(u)/u^p$ is decreasing*) *or equivalently $\varphi(u^{1/p})$ is starshaped* (*respectively reversed starshaped*), *then $L_\varphi$ satisfies an upper $p$-estimate* (*respectively a lower $p$-estimate*) *with constant one.*

(ii) *Let $L_\varphi$ satisfy an upper $p$-estimate* (*respectively a lower $p$-estimate*) *with constant one. If $\mu T = \infty$, then the function $\varphi(u)/u^p$ is pseudo-increasing with constant* 2 (*respectively pseudo-decreasing with constant* 2), *that is*

$$\frac{\varphi(t)}{t^p} \leqslant 2\frac{\varphi(s)}{s^p} \quad \left(\textit{respectively } \frac{\varphi(t)}{t^p} \geqslant 2\frac{\varphi(s)}{s^p}\right) \quad \textit{for } 0 < t \leqslant s.$$

*If $\mu T < \infty$ then the above inequalities are satisfied for all $s \geqslant t \geqslant (\varphi^{-1}(2/\mu T))^p/2$.*

**Proof.** Part (i) follows immediately from Corollary 3.3. We provide the proof of part (ii) only in the case when $p = 1$ and $\mu T < \infty$. Let $L_\varphi$ satisfy an upper 1-estimate with constant one. By Theorem 3.2, $\varphi$ satisfies (C) for all $u, v > 0$ with $1/\varphi(u) + 1/\varphi(v) \leqslant \mu T$. It follows the "restricted" superadditivity, that is $\varphi(u) + \varphi(v) \leqslant \varphi(u + v)$ for all $u, v > 0$ with $\varphi(u + v) \geqslant 2/\mu T$. Thus by induction we obtain for $n \in \mathbb{N}$,

$$\frac{\varphi(nu)}{n} \geqslant \varphi(u) \quad \text{if } u \geqslant \varphi^{-1}(2/\mu T)/2,$$

and so

$$\frac{\varphi(v)}{n} \geqslant \varphi\left(\frac{v}{n}\right) \quad \text{whenever } v/n \geqslant \varphi^{-1}(2/\mu T)/2.$$

Let $\varphi^{-1}(2/\mu T)/2 \leqslant t < s$ and put $a = t/s$. Then $a \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for some $n \in \mathbb{N}$. Hence $\varphi^{-1}(2/\mu T)/2 \leqslant t \leqslant s/n$ and so

$$\varphi(t) = \varphi(as) \leqslant \varphi\left(\frac{s}{n}\right) \leqslant \frac{\varphi(s)}{n} \leqslant 2a\varphi(s) = 2t\frac{\varphi(s)}{s}. \qquad \square$$

For some classes of Orlicz functions $\varphi$, in particular such that $\varphi(u^{1/p})$ is "close" to a straight line in a neighbourhood of zero or infinity, starshapeness is also a necessary condition for one-$p$-estimation of $L_\varphi$.

**Proposition 3.5.** *Let $0 < p < \infty$ and $L_\varphi$ be an Orlicz space defined on a non-atomic infinite (respectively finite) measure space. Assume*

$$\sup_{u>0} \frac{\varphi((au)^{1/p})}{a\varphi(u^{1/p})} \geqslant 1, \quad \textit{for every } a \in (0,1),$$
$$\left(\textit{respectively } \sup_{u>u_0} \frac{\varphi((au)^{1/p})}{a\varphi(u^{1/p})} \geqslant 1 \quad \textit{for some } u_0 > 0 \textit{ and all } a \in (0,1)\right).$$

*If $L_\varphi$ has an upper $p$-estimate with constant one, then $\varphi(u^{1/p})$ is starshaped, i.e., $\varphi(u)/u^p$ is increasing (respectively $\varphi(u)/u^p$ is increasing on $[v_0, \infty)$, where $v_0$ is any number satisfying $1/\varphi(u_0^{1/p}) + 1/\varphi(v_0^{1/p}) \leqslant \mu T$).*

**Proof.** We give a proof only in the case of finite measure and $p = 1$. Assume that $\varphi(u)/u$ is not increasing on some $[v_0, \infty)$ such that $1/\varphi(u_0) + 1/\varphi(v_0) \leqslant \mu T$. Then there exist $0 < a < 1$, $K > 1$ and $v > v_0$ such that

$$\varphi(av) > Ka\varphi(v).$$

Applying the assumption for $1 - a$ (instead of $a$), there exist $\delta \in (0, 1)$ and $u > u_0$ such that

$$\varphi\big((1-a)u\big) > (1-\delta)(1-a)\varphi(u).$$

Clearly we can assume that $\delta < \frac{a(K-1)}{1-a}$. Let $b = \frac{1-a}{a}$. Since $1/\varphi(u) + 1/\varphi(v) \leqslant 1/\varphi(u_0) + 1/\varphi(v_0) \leqslant \mu T$, we can choose disjoint sets $A$ and $B$ such that $\mu A = 1/\varphi(u)$ and $\mu B = 1/\varphi(v)$. Then setting

$$f = v\chi_B \quad \text{and} \quad g = bu\chi_A,$$

we have $\|f\|_\varphi = 1$ and $\|g\|_\varphi = b$. Thus

$$\begin{aligned}
I_\varphi\left(\frac{f+g}{1+b}\right) &= I_\varphi\left(\frac{f}{1+b}\right) + I_\varphi\left(\frac{g}{1+b}\right) \\
&= I_\varphi(af) + I_\varphi\left((1-a)\frac{g}{b}\right) \\
&= \varphi(av)\mu B + \varphi\big((1-a)u\big)\mu A \\
&> Ka\varphi(v)\mu B + (1-\delta)(1-a)\varphi(u)\mu A \\
&= Ka + (1-\delta)(1-a) = 1 + (K-1)a - \delta(1-a) \\
&> 1 + \frac{1-a}{a}a\delta - \delta(1-a) = 1.
\end{aligned}$$

Therefore $\|f + g\|_\varphi > 1 + b = \|f\|_\varphi + \|g\|_\varphi$, and so $L_\varphi$ does not satisfy an upper 1-estimate with constant one. □

The counterpart of Proposition 3.5 for lower $p$-estimate is stated as follows and can be proved in a similar manner.

**Proposition 3.6.** *Let $0 < p < \infty$ and $L_\varphi$ be an Orlicz space defined on a non-atomic infinite (respectively finite) measure space. Assume*

$$\inf_{u>0} \frac{\varphi((au)^{1/p})}{a\varphi(u^{1/p})} \leqslant 1, \quad \text{for every } a \in (0,1),$$
$$\left(\text{respectively } \inf_{u>u_0} \frac{\varphi((au)^{1/p})}{a\varphi(u^{1/p})} \leqslant 1 \quad \text{for some } u_0 > 0 \text{ and all } a \in (0,1)\right).$$

*If $L_\varphi$ has a lower $p$-estimate with constant one, then $\varphi(u^{1/p})$ is reversed starshaped, i.e., $\varphi(u)/u^p$ is decreasing (respectively $\varphi(u)/u^p$ is decreasing on $[v_0, \infty)$, where $v_0$ is any number satisfying $1/\varphi(u_0^{1/p}) + 1/\varphi(v_0^{1/p}) \leqslant \mu T$).*

In the next example we shall show some applications of Proposition 3.5.

**Example 3.7.** Let

$$\varphi(u) = (n-1) + \big(u - (n-1)\big)^2 \quad \text{for } u \in [n-1, n],\ n \in \mathbb{N}.$$

Then $L_\varphi$ does not satisfy the upper 1-estimate with constant one on any non-atomic measure space $T$ such that $1 < \mu T \leqslant \infty$.

In the case of infinite measure this follows from Theorem 3.1, since $\varphi$ does not satisfy condition (C) (compare with Example 2.1). If $\mu T < \infty$, then we apply Proposition 3.5. Indeed, for every $a = m/n \in (0,1), m, n \in \mathbb{N}$ and $u = n$ we have $\varphi(au) = a\varphi(u)$, and so for any $a \in (0,1)$,

$$\sup_{u \geqslant 1} \frac{\varphi(au)}{a\varphi(u)} \geqslant 1.$$

Hence the assumption of Proposition 3.5 is satisfied for $u_0 = 1$. Consequently, if $L_\varphi$ satisfied the upper 1-estimate with constant one, then $\varphi(u)/u$ would be increasing on any $[v, \infty)$ for every $v$ with $1 + 1/\varphi(v) \leqslant \mu T$. However, this is impossible since we can easily show that $\varphi(u)/u$ is not increasing on any interval $[v, \infty)$, $v > 0$.

We shall finish this section with criteria of one-upper and lower $p$-estimate of the sequence space $\ell_\varphi$.

**Theorem 3.8.** *Let $0 < p < \infty$ and $\varphi$ be an Orlicz function. The sequence Orlicz space $\ell_\varphi$ satisfies the upper $p$-estimate (respectively lower $p$-estimate) with constant one if and only if for every $a, u_i, v_i \in [0,1]$, $i = 1, \dots, n, n \in \mathbb{N}$, such that $\sum_{i=1}^n \varphi(u_i^{1/p}) = \sum_{i=1}^n \varphi(v_i^{1/p}) = 1$, we have*

$$\sum_{i=1}^n \left(\varphi\big((au_i)^{1/p}\big) + \varphi\Big(\big((1-a)v_i\big)^{1/p}\Big)\right) \leqslant 1,$$
$$\left(\text{respectively } \sum_{i=1}^n \left(\varphi\big((au_i)^{1/p}\big) + \varphi\Big(\big((1-a)v_i\big)^{1/p}\Big)\right) \geqslant 1\right).$$

**Proof.** We shall consider only upper estimation for $p = 1$. Assume the inequality in the hypothesis holds and take $x = \sum_{i\in A} x_i e_i$ and $y = \sum_{j\in B} y_j e_j$, where $A, B$ are disjoint, finite subsets of natural numbers, $e_i$ are the unit vectors, and $x_i$, $y_i$ are real numbers. Setting $\alpha = \|x\|_\varphi$ and $\beta = \|y\|_\varphi$, we get

$$I_\varphi\left(\frac{x+y}{\alpha+\beta}\right) = \sum_{i\in A}\varphi\left(\frac{\alpha}{\alpha+\beta}\frac{|x_i|}{\alpha}\right) + \sum_{i\in B}\varphi\left(\frac{\beta}{\alpha+\beta}\frac{|y_i|}{\beta}\right) \leqslant 1,$$

since $\sum_{i\in A}\varphi(|x_i|/\alpha) = \sum_{i\in B}\varphi(|y_i|/\beta) = 1$. It follows that $\|x+y\|_\varphi \leqslant \|x\|_\varphi + \|y\|_\varphi$. By the limit argument and the Fatou property we obtain the triangle inequality for arbitrary disjoint sequences $x$ and $y$.

Now suppose the inequality in the hypothesis is not satisfied. Thus there exist $n \in \mathbb{N}$, $a, u_i, v_i \in [0, 1]$, $i = 1, \ldots, n$ such that $\sum_{i=1}^n \varphi(u_i) = \sum_{i=1}^n \varphi(v_i) = 1$ and

$$\sum_{i=1}^n \big(\varphi(au_i) + \varphi\big((1-a)v_i\big)\big) > 1.$$

Let $\alpha = a$ and $\beta = 1 - a$ and set

$$x = \sum_{i=1}^n \alpha u_i e_i \quad \text{and} \quad y = \sum_{i=n+1}^{2n} \beta v_{i-n} e_i.$$

Then $x$ and $y$ have disjoint supports and $\|x/\alpha\|_\varphi = \sum_{i=1}^n \varphi(u_i) = 1$ and $\|y/\beta\|_\varphi = \sum_{i=1}^n \varphi(v_i) = 1$. Moreover,

$$I_\varphi\left(\frac{x+y}{\alpha+\beta}\right) = \sum_{i=1}^n \big(\varphi(au_i) + \varphi\big((1-a)v_i\big)\big) > 1,$$

which implies $\|x+y\|_\varphi > \alpha + \beta = \|x\|_\varphi + \|y\|_\varphi$. $\square$

Applying the above theorem and the method of proof of Theorem 3.1 we quickly get the last result of this section, which states among others that in the sequence space $\ell_\varphi$, it is sufficient to know the behaviour of $\varphi$ only in the interval $(0, 1)$.

**Corollary 3.9.** *Let $0 < p < \infty$. If $\varphi(u^{1/p})$ satisfies condition* (C) (*respectively* (RC)) *for every $u, v \in (0, 1)$ then $\ell_\varphi$ satisfies an upper $p$-estimate* (*respectively lower $p$-estimate*) *with constant one. Conversely, if $\ell_\varphi$ satisfies an upper $p$-estimate* (*respectively lower $p$-estimate*) *with constant one, then $\varphi(u^{1/p})$ satisfies condition* (C) (*respectively* (RC)) *for every $u, v \in (0, 1)$ such that $1/\varphi(u^{1/p})$ and $1/\varphi(v^{1/p})$ are natural numbers.*

## 4. Dilation-like operations

We now introduce dilation-like operations that are closely related to lower and upper $p$-estimates, and which can be defined in a general context of rearrangement invariant spaces. Let $X$ be a rearrangement invariant function space [7] over $(T, \Sigma, \mu)$, and let $f \in X$. We consider vectors that generalize sums of many disjointly supported copies of $f$. For a natural number $m \geqslant 1$ by $f^{(m)}$ we denote any function which is a sum of $m$ disjointly supported functions $f_1 + \cdots + f_m$ such that $f_i$ has the same distribution as $f$ for $i = 1, \dots, m$, that is, $\mu\{|f| > \lambda\} = \mu\{|f_i| > \lambda\}$, $\lambda > 0$. If $\mu T < \infty$, then we define $f^{(m)}$ for $f$'s whose support has measure less than or equal to $\mu T/m$ only. If $g = f^{(m)}$ then we write $f = g^{(1/m)}$. Finally, if $g = f^{(m)}$ and $\xi = n/m$ is a positive rational number then by $g^{(\xi)}$ we denote any vector of the form $f^{(n)}$. In this convention $f^{(1)}$ is any vector with the same distribution as $f$. Moreover, if for example, $\mu T = \infty$ and $f = \chi_B$ for some set $B$ of finite measure, then $f^{(m)} = \chi_{A'}$ where $A'$ is any measurable set with $\mu A' = m\mu B$, and more generally, for rational $\xi$, $f^{(\xi)} = \chi_A$ where $A$ is any measurable set with $\mu A = \xi \mu B$. (If $\mu T < \infty$ then we need $\mu B \leqslant \mu T/\xi$, in order for $f^{(\xi)}$ to be defined.) The operation $f^{(\lambda)}$ has an obvious connection with dilation operator if $T = [0, \infty)$. In fact, if $f : \mathbb{R}_+ \to \mathbb{R}$ is any function, then the $\lambda$-dilation of $f$ is defined as the function $f(t/\lambda)$, $t \geqslant 0$. It is clear that for rational $\lambda$, $f(t/\lambda)$ can be treated as $f^{(\lambda)}$.

Let us recall a variant of upper and lower $p$-estimates that requires that appropriate inequalities hold for (disjointly supported) vectors which have equal norms. This is obviously weaker than the appropriate upper and lower $p$-estimates. However, due to this slightly simplified structure it admits, in the case of Orlicz spaces, a characterization in terms of much simpler properties of $\varphi$, as well it has an interesting connection to the dilation-like operation $f^{(n)}$, $n \in \mathbb{N}$.

**Theorem 4.1.** *Let $0 < p < \infty$, let $\varphi$ be an Orlicz function, and let $L_\varphi$ be the corresponding Orlicz space on a non-atomic measure space. Consider the following conditions*:

(i) $\varphi(u) \leqslant \varphi(n^{1/p}u)/n$, *for all* $n = 1, 2, \dots$ *and* $u > 0$;
(i-0) $\varphi(u) \leqslant \varphi(n^{1/p}u)/n$, *for* $u > \varphi^{-1}(1/\mu T)$;
(ii) $\|f^{(n)}\|_\varphi \leqslant n^{1/p}\|f\|_\varphi$, *for all* $n = 1, 2, \dots$ *and* $f \in L_\varphi$ *for which* $f^{(n)}$ *is defined*;
(iii) $L_\varphi$ *satisfies the upper* $p$*-estimate with constant one for equal norm vectors.*

*Then* (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i-0). *In particular, if* $\mu T = \infty$ *then all conditions are equivalent.*

**Proof.** (i) $\Rightarrow$ (iii). As usual we can assume that $p = 1$. Then for all $n = 1, 2, \dots$ and $w > 0$ we have

$$\varphi(w/n) \leqslant (1/n)\varphi(w).$$

Let $f_i$, $i = 1, \dots, n$, be disjointly supported functions in $L_\varphi$, with $\|f_i\|_\varphi = \|f_1\|_\varphi$, and pick an arbitrary $a > \|f_1\|_\varphi$. Then $I_\varphi(f_i/a) \leqslant 1$. Moreover,

$$I_\varphi\left(\frac{1}{na}\sum_{i=1}^n f_i\right) = \sum_{i=1}^n I_\varphi\left(\frac{1}{n}\frac{f_i}{a}\right) \leqslant \sum_{i=1}^n \frac{1}{n} I_\varphi(f_i/a) \leqslant 1.$$

Hence

$$\left\| \sum_{i=1}^n f_i \right\|_\varphi \leqslant na,$$

and since $a > \|f_i\|_\varphi$ is arbitrary, we get $\|\sum_{i=1}^n f_i\|_\varphi \leqslant n\|f_1\|_\varphi$, as required.

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i-0). Fix any $u > \varphi^{-1}(1/\mu T)$ and $n \geqslant 1$. There exist measurable sets $A$ and $B$ with $\mu A = 1/(n\varphi(u))$ and $\mu B = 1/\varphi(u)$. Let $f = u\chi_A$ and $h = u\chi_B$. Then $f^{(n)}$ is defined and equal to $h$. Note that since $I_\varphi(h) = 1$ then $\|h\|_\varphi = 1$. Thus the condition $\|h\|_\varphi \leqslant n^{1/p}\|f\|_\varphi$ is the same as $\|f\|_\varphi \geqslant 1/n^{1/p}$. This in turn is equivalent to $I_\varphi(n^{1/p} f) \geqslant 1$ which means, by the definitions of $f$ and $A$, that $1 \leqslant \varphi(n^{1/p}u)\mu A = \varphi(n^{1/p}u)/n\varphi(u)$, or equivalently, that $\varphi(n^{1/p}u)/n \geqslant \varphi(u)$, and (i-0) is satisfied.

The equivalence of the conditions in the case of an infinite measure is obvious, as $\varphi^{-1}(0) = 0$. $\square$

A counterpart of Theorem 4.1 stated below follows by an analogous argument.

**Theorem 4.2.** *Let $0 < p < \infty$, let $\varphi$ be an Orlicz function, and let $L_\varphi$ be the corresponding Orlicz space on a non-atomic measure space. Consider the following conditions*:

(i) $\varphi(u) \geqslant \varphi(n^{1/p}u)/n$, *for all* $n = 1, 2, \ldots$ *and* $u > 0$;
(i-0) $\varphi(u) \geqslant \varphi(n^{1/p}u)/n$, *for* $u > \varphi^{-1}(1/\mu T)$;
(ii) $\|f^{(n)}\|_\varphi \geqslant n^{1/p}\|f\|_\varphi$, *for all* $n = 1, 2, \ldots$ *and* $f \in L_\varphi$ *for which* $f^{(n)}$ *is defined*;
(iii) $L_\varphi$ *satisfies the lower $p$-estimate with constant one for equal norm vectors.*

*Then* (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i-0). *In particular, if $\mu T = \infty$, then all conditions are equivalent.*

As we have seen in Section 3, the starshapeness and reversed starshapeness of $\varphi(u^{1/p})$, are in general too strong for being necessary conditions for one-upper and lower $p$-estimates of $L_\varphi$. However, as we shall see in the next two results, they have interesting interpretations in terms of some "boundedness inequalities" for $\|f\|_\varphi$ and $\|f^{(\lambda)}\|_\varphi$.

**Theorem 4.3.** *Let $0 < p < \infty$, and let $L_\varphi$ be the corresponding Orlicz space on an non-atomic measure space. Consider the following conditions*:

(i) $\varphi(u)/u^p$ *is increasing on* $(0, \infty)$, *that is* $\varphi(u^{1/p})$ *is starshaped*;
(i-0′) $\varphi(u)/u^p$ *is increasing on* $(\varphi^{-1}(1/\mu T), \infty)$;
(ii) $\|f^{(\lambda)}\|_\varphi \leqslant \lambda^{1/p}\|f\|_\varphi$, *for all rational* $\lambda \geqslant 1$ *and* $f \in L_\varphi$.

*Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (i-0′). *In particular, if $\mu T = \infty$ then* (i) *and* (ii) *are equivalent.*

**Proof.** (ii) $\Rightarrow$ (i-0′). We follow the same argument as in Theorem 4.1, by choosing, for a fixed $u > \varphi^{-1}(1/\mu T)$ and rational $\lambda \geqslant 1$, sets $A$ and $B$ with $\mu A = 1/\lambda\varphi(u)$ and $\mu B = 1/\varphi(u)$, and letting $f = u\chi_A$ and $h = u\chi_B$, as before. Then $h = f^{(\lambda)}$ and $\|h\|_\varphi = 1$, and the same argument as before shows that $\varphi(\lambda^{1/p}u)/\lambda \geqslant \varphi(u)$. Since this is true for all rational $\lambda \geqslant 1$, by continuity the same is true for all real $\lambda \geqslant 1$. Thus $\varphi(\lambda u)/\lambda^p \geqslant \varphi(u)$ for all $\lambda \geqslant 1$ and all $u > \varphi^{-1}(1/\mu T)$, which is equivalent to $\varphi(u)/u^p$ being increasing on $(\varphi^{-1}(1/\mu T), \infty)$.

(i) $\Rightarrow$ (ii). First, let $f$ be of the form $f = u\chi_A$ for some $u > 0$ and a subset $A$, let $\lambda \geqslant 1$ be rational and assume that $\|f^{(\lambda)}\|_\varphi = 1$. Then noting that the implications in the first part of the proof are in fact equivalences, (i) implies that $1/\lambda^{1/p} \leqslant \|f\|_\varphi$. By homogeneity this implies that (ii) is true for any $f = u\chi_A$.

Next, let $f$ be a simple function, for example, $f = u_1\chi_{A_1} + u_2\chi_{A_2}$ with $A_1 \cap A_2 = \emptyset$ (the general case is obviously the same). Let $h = u_1\chi_{B_1} + u_2\chi_{B_2}$, where $B_1 \cap B_2 = \emptyset$ and $\mu B_i = \lambda\mu A_i$ for $i = 1, 2$, so that $h = f^{(\lambda)}$. Assume that $\|h\|_\varphi = 1$, that is, $\varphi(u_1)\mu B_1 + \varphi(u_2)\mu B_2 = 1$. Then by (i) and the normalization above,

$$I_\varphi\big(\lambda^{1/p} f\big) = \varphi\big(\lambda^{1/p}u_1\big)\mu A_1 + \varphi\big(\lambda^{1/p}u_2\big)\mu A_2 \geqslant 1.$$

In view of the Fatou property by the limit argument we obtain the above inequality for any $f \in L_\varphi$. This means $\|f\|_\varphi \geqslant 1/\lambda^{1/p}$, which by homogeneity, implies (ii). $\square$

A counterpart of Theorem 4.3 says:

**Theorem 4.4.** *Let $0 < p < \infty$, let $\varphi$ be an Orlicz function, and let $L_\varphi$ be the corresponding Orlicz space on a non-atomic measure space. Consider the following conditions*:

(i) $\varphi(u)/u^p$ *is decreasing on* $(0, \infty)$*, that is* $\varphi(u^{1/p})$ *is reversed starshaped*;
(i-0) $\varphi(u)/u^p$ *is decreasing on* $(\varphi^{-1}(1/\mu T), \infty)$;
(ii) $\|f^{(\lambda)}\|_\varphi \geqslant \lambda^{1/p}\|f\|_\varphi$*, for all rational* $\lambda \geqslant 1$ *and* $f \in L_\varphi$ *for which* $f^{(\lambda)}$ *is defined.*

*Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (i-0)*. In particular, if* $\mu T = \infty$ *then* (i) *and* (ii) *are equivalent.*

## 5. $p$-Convexity and $p$-concavity

We start with a criterion on $p$-convexity ($p$-concavity) of Orlicz function space $L_\varphi$ over a non-atomic measure space.

**Theorem 5.1.** *Let $0 < p < \infty$ and $\varphi$ be an Orlicz function. Then the Orlicz space $L_\varphi$ on a non-atomic measure space is $p$-convex* (*respectively $p$-concave*) *with constant one if and only if $\varphi(u^{1/p})$ is convex* (*respectively concave*).

**Proof.** We shall give a proof only for convexity when $p = 1$. It is well known that if $\varphi$ is convex, then $\|\cdot\|_\varphi$ satisfies the triangle inequality, that is 1-convexity with constant one, since it is the Minkowski's functional of the convex set $\{f\colon\ I_\varphi(f) \leqslant 1\}$.

Assume now that $\varphi$ is not convex. Then there exist $u_1, u_2 > 0$ such that

$$\varphi\left(\frac{u_1 + u_2}{2}\right) > \frac{\varphi(u_1) + \varphi(u_2)}{2}.$$

Let first $\mu T = \infty$. Choose two disjoint measurable sets $A_1, A_2$ such that $\mu A_1 = \mu A_2$ and

$$\varphi(u_1)\mu A_1 + \varphi(u_2)\mu A_2 = 1.$$

Let $f_1 = u_1\chi_{A_1} + u_2\chi_{A_2}$ and $f_2 = u_1\chi_{A_2} + u_2\chi_{A_1}$. Then $I_\varphi(f_1) = I_\varphi(f_2) = 1$ and hence $\|f_1\|_\varphi = \|f_2\|_\varphi = 1$. Moreover,

$$I_\varphi\left(\frac{f_1 + f_2}{2}\right) = \varphi\left(\frac{u_1 + u_2}{2}\right)\mu A_1 + \varphi\left(\frac{u_1 + u_2}{2}\right)\mu A_2 > 1,$$

which implies that $\|f_1 + f_2\|_\varphi > 2 = \|f_1\|_\varphi + \|f_2\|_\varphi$.

Now let $\mu T < \infty$. We first choose two disjoint sets $A_i$, $i = 1, 2$, such that $\mu A_1 = \mu A_2$ and

$$\varphi(u_1)\mu A_1 + \varphi(u_2)\mu A_2 := \alpha < 1.$$

Then we choose a measurable set $C \subset T \setminus (A_1 \cup A_2)$ with $\mu C > 0$ and subsequently $u > 0$ such that $\varphi(u)\mu C = 1 - \alpha$. Setting

$$f_1 = u_1\chi_{A_1} + u_2\chi_{A_2} + u\chi_C \quad \text{and} \quad f_2 = u_2\chi_{A_1} + u_1\chi_{A_2} + u\chi_C,$$

we have $\|f_i\|_\varphi = 1$, $i = 1, 2$, and analogously as above $\|f_1 + f_2\|_\varphi > 2$, which completes the proof. $\square$

We now pass to the $p$-convexity and $p$-concavity properties for Orlicz sequence spaces $\ell_\varphi$.

**Theorem 5.2.** *Let $0 < p < \infty$ and $\varphi$ be an Orlicz function. If $\varphi(u^{1/p})$ is convex (respectively concave) on $(0, 1)$, then $\ell_\varphi$ or $\ell_\varphi^m$ is $p$-convex (respectively $p$-concave) with constant one. We have a partial converse: if $\ell_\varphi$ or $\ell_\varphi^m$ with $m \geqslant 3$ is $p$-convex (respectively $p$-concave) with constant one then $\varphi(u^{1/p})$ is convex (respectively concave) on $(0, (\varphi^{-1}(1/2))^p)$.*

Before passing to the proof of the theorem let us note that the assumption that the dimension of the Orlicz space is larger than 2 is essential: the Minkowski functional of a two-dimensional Orlicz space can be a norm, while the function $\varphi$ does not need to be convex on the interval $(0, 1)$ (or even on a smaller interval).

**Example 5.3.** There exists an Orlicz function $\varphi$, which is not convex on $(0, 1/2) = (0, \varphi^{-1}(1/2))$, but $\|\cdot\|_\varphi$ is a norm on $\mathbb{R}^2$.

Let

$$\varphi(x) = \frac{1}{2}(2x - 1)^3 + \frac{1}{2}, \quad 0 \leqslant x \leqslant 1.$$

It is easy to check that $\varphi$ is an Orlicz function not convex on $(0, 1/2)$. However, the curve $\varphi(x) + \varphi(y) = 1$ is equivalent to $x + y = 1$ for $x, y \in [0, 1]$. Therefore the set $\{(x, y)\colon \varphi(|x|) + \varphi(|y|) \leqslant 1\}$ is the unit ball in the space $\ell_1^2$, and thus $\|\cdot\|_\varphi$ satisfies the triangle inequality.

**Proof of Theorem 5.2.** It is easy to check that the set $\{x\colon I_\varphi(x) \leqslant 1\}$ is convex whenever $\varphi$ is convex on $(0, 1)$, and thus $\|\cdot\|_\varphi$ satisfies the triangle inequality as a Minkowski's functional.

To prove the (partial) converse statement, again it is enough to consider the case $p = 1$. Assume to the contrary that $\varphi$ is not convex on the interval $(0, \varphi^{-1}(1/2))$. Thus there exist $0 < a < b < \varphi^{-1}(1/2)$ such that

$$\varphi\left(\frac{a+b}{2}\right) > \frac{\varphi(a)+\varphi(b)}{2}.$$

Let $x = ae_1 + be_2 + ce_3$ and $y = be_1 + ae_2 + ce_3$, where $c > 0$ is such that $\varphi(a) + \varphi(b) + \varphi(c) = 1$. Hence $\|x\|_\varphi = \|y\|_\varphi = 1$ and

$$I_\varphi\left(\frac{x+y}{2}\right) = 2\varphi\left(\frac{a+b}{2}\right) + \varphi(c) > \varphi(a) + \varphi(b) + \varphi(c) = 1.$$

Therefore $\|x + y\|_\varphi > 2$. $\square$

For some functions $\varphi$ and the infinite-dimensional spaces $\ell_\varphi$, we obtain the full converse of Theorem 5.2. The situation is in a sense analogous to Propositions 3.5 and 3.6. Let us consider first the case $p = 1$.

**Lemma 5.4.**

(i) *Let $\varphi$ be an Orlicz function such that*

$$\limsup_{u\to 0} \frac{2\varphi(u)}{\varphi(2u)} \geqslant 1.$$

*If $\ell_\varphi$ is 1-convex with constant one, then $\varphi$ is convex on $(0, 1)$.*

(ii) *Let $\varphi$ be an Orlicz function such that*

$$\liminf_{u\to 0} \frac{2\varphi(u)}{\varphi(2u)} \leqslant 1.$$

*If $\ell_\varphi$ is 1-concave with constant one, then $\varphi$ is concave on $(0, 1)$.*

**Proof.** (i) Let $\varphi$ be not convex on $(0, 1)$. Then there exist $0 < x_1 < y_1 < 1$ such that

$$\varphi\left(\frac{x_1+y_1}{2}\right) \geqslant K\frac{\varphi(x_1)+\varphi(y_1)}{2},$$

where $K > 1$. Let

$$a = (K-1)\frac{\varphi(x_1)+\varphi(y_1)}{2}$$

and $\alpha = \varphi(y_1) - \varphi(x_1)$. Without loss of generality we can assume that $\alpha > a$. By the assumptions there exist $0 < u_n \downarrow 0$, $0 < \delta_n \downarrow 0$ such that for all $n \in \mathbb{N}$,

$$\varphi\left(\frac{u_n}{2}\right) > \frac{1-\delta_n}{2}\varphi(u_n),$$

where $\delta_1 < a/\alpha$ and $\varphi(u_1) < \alpha$. Since $\varphi(u_n) \to 0$, we can choose $k, m \in \mathbb{N}$ such that

$$\alpha - a/2 \leqslant \varphi(u_k)m \leqslant \alpha.$$

Since $y_1 < 1$, we can choose $y_2 > 0$ such that

$$\varphi(y_1) + \varphi(y_2) = 1.$$

Then setting

$$y = y_1 e_1 + y_2 e_2$$

we have $I_\varphi(y) = \|y\|_\varphi = 1$. We then let $x_2 = y_2$ and set

$$x = x_1 e_1 + x_2 e_2 + u_k(e_3 + \cdots + e_{m+2}).$$

Thus

$$I_\varphi(x) = \varphi(x_1) + \varphi(x_2) + \varphi(u_k)m = 1 - \alpha + \varphi(u_k)m.$$

Hence by the choice of $k$ and $m$,

$$1 \geqslant I_\varphi(x) \geqslant 1 - a/2.$$

It follows that $\|x\|_\varphi \leqslant 1$, and

$$\begin{aligned} I_\varphi\left(\frac{x+y}{2}\right) &\geqslant K\frac{\varphi(x_1)+\varphi(y_1)}{2} + \frac{\varphi(x_2)}{2} + \frac{\varphi(y_2)}{2} + \frac{1-\delta_k}{2} m\varphi(u_k) \\ &= \frac{1}{2}\bigl(\varphi(y_1)+\varphi(y_2)\bigr) + a + \frac{1}{2}\bigl(\varphi(x_1)+\varphi(x_2)+m\varphi(u_k)\bigr) - \frac{\delta_k}{2} m\varphi(u_k) \\ &\geqslant \frac{1}{2} + a + \frac{1}{2}\left(1 - \frac{a}{2}\right) - \frac{\delta_k}{2} m\varphi(u_k) \geqslant 1 + \frac{3}{4}a - \frac{\delta_k}{2}\alpha \\ &\geqslant 1 + \frac{3}{4}a - \frac{a}{2} = 1 + \frac{1}{4}a > 1. \end{aligned}$$

Hence we have $\|x+y\|_\varphi > 2$, which contradicts the triangle inequality $\|x+y\|_\varphi \leqslant \|x\|_\varphi + \|y\|_\varphi \leqslant 2$.

(ii) The proof is similar. If $\varphi$ is not concave on $(0, 1)$ then there exist $0 < x_1 < y_1 < 1$ and $0 < K < 1$ such that

$$\varphi\left(\frac{x_1+y_1}{2}\right) \leqslant K\frac{\varphi(x_1)+\varphi(y_1)}{2}.$$

Setting $a = (1-K)(\varphi(x_1)+\varphi(x_2))/2$ and $\alpha = \varphi(y_1) - \varphi(x_1)$ we assume that $\alpha > a$. There exist $0 \leqslant u_n \downarrow 0$, $0 < \delta_n \downarrow 0$ such that for all $n \in \mathbb{N}$,

$$\varphi\left(\frac{u_n}{2}\right) \leqslant \frac{1+\delta_n}{2}\varphi(u_n),$$

where $\delta_1 < a/\alpha$ and $\varphi(u_1) < \alpha$. We choose $k, m \in \mathbb{N}$ such that

$$\alpha \leqslant \varphi(u_k)m \leqslant \alpha + a/2.$$

Define $x$ and $y$ as in part (i). Then $I_\varphi(y) = \|y\|_\varphi = 1$ and $1 \leqslant I_\varphi(x) \leqslant 1 + a/2$. Hence

$$I_\varphi\left(\frac{x+y}{2}\right) \leqslant \frac{\varphi(y_1)}{2} + \frac{\varphi(y_2)}{2} - a + \frac{\varphi(x_1)}{2} + \frac{\varphi(x_2)}{2} + m\varphi\left(\frac{u_k}{2}\right) \leqslant 1 - \frac{a}{4} < 1,$$

and so $\|x+y\|_\varphi < 2$. It contradicts the inequality $\|x+y\|_\varphi \geqslant \|x\|_\varphi + \|y\|_\varphi \geqslant 2$ and completes the proof. $\square$

In view of the above lemma and Theorem 5.2 we get the following result.

**Theorem 5.5.** *Let $0 < p < \infty$ and $\varphi$ be an Orlicz function such that*

$$\limsup_{u\to 0} \frac{2\varphi(u^{1/p})}{\varphi((2u)^{1/p})} \geqslant 1 \quad \left(\textit{respectively } \liminf_{u\to 0} \frac{2\varphi(u^{1/p})}{\varphi((2u)^{1/p})} \leqslant 1\right).$$

*Then $\ell_\varphi$ is $p$-convex* (*respectively $p$-concave*) *with constant one if and only if $\varphi(u^{1/p})$ is convex* (*respectively concave*) *on* $(0, 1)$.

We finish with applications on moduli of convexity and smoothness in Orlicz spaces. For the definitions of these moduli we refer the reader to [7, Vol. II].

**Corollary 5.6.** *Assume that $\varphi$ is a convex Orlicz function and $L_\varphi$ is an Orlicz space over a non-atomic measure space. Let $1 < p < 2 < q < \infty$.*

(i) *If $\varphi(u^{1/p})$ is convex and $\varphi(u^{1/q})$ satisfies condition* (RC)*, then $L_\varphi$ has modulus of convexity of power type $q$.*
(ii) *If $\varphi(u^{1/q})$ is concave and $\varphi(u^{1/p})$ satisfies condition* (C)*, then $L_\varphi$ has modulus of smoothness of power type $p$.*
(iii) *If $\varphi(u^{1/p})$ is convex and $\varphi(u^{1/q})$ is concave, where $1 < p \leqslant 2 \leqslant q < \infty$, then $L_\varphi$ has modulus of convexity of power type $q$ and modulus of smoothness of power type $p$.*

**Proof.** The proof follows by combining general facts on Banach lattices and the results of this paper. Condition (i) is a consequence of the proof of Theorem 1.f.10 in [7, vol. II], and Theorem 5.1 and Corollary 3.3. Condition (ii) follows from (i) by duality. Indeed, a well-known argument (cf. [7, vol. II, Proposition 1.d.4]) shows that a Banach lattice $X$ is one-$q$-concave if and only if $X^*$ is one-$q'$-convex (where $q' = q/(q-1)$). Analogous duality facts hold for upper and lower estimates as well as for moduli of convexity and smoothness. Condition (iii) follows immediately from Theorem 5.1 and Theorem 1.f.1 in [7, vol. II]. □

Recall that complex Orlicz spaces $L_\varphi$ are sets of those $\Sigma$-measurable complex-valued functions $f$ on $T$ for which $|f| \in L_\varphi$. We also notice that all results included in this paper are valid for complex Orlicz spaces as well. The next result on the modulus of complex convexity of $L_\varphi$ is an immediate consequence of [1, Theorem 7.3] and Corollary 3.3.

**Corollary 5.7.** *Let $L_\varphi$ be a complex Orlicz space over a non-atomic measure space. If $2 < q < \infty$ and $\varphi(u^{1/q})$ satisfies condition* (RC)*, then $L_\varphi$ is $q$-uniformly PL-convex.*

In view of Corollary 3.9 and Theorem 5.2, the above Corollaries 5.6 and 5.7 remain valid also for sequence spaces $\ell_\varphi$. In fact, the conditions on $\varphi(u^{1/p})$ and $\varphi(u^{1/q})$ can be restricted to the interval (0, 1) only.

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